Renormalization group and nonequilibrium action in stochastic field theory

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We investigate the renormalization group approach to nonequilibrium field theory. We show that it is possible to derive nontrivial renormalization group flow from iterative coarse graining of a closed-time-path action. This renormalization group is different from the usual in quantum field theory textbooks, in that it describes nontrivial noise and dissipation. We work out a specific example where the variation of the closed-time-path action leads to the so-called Kardar-Parisi-Zhang equation, and show that the renormalization group obtained by coarse graining this action, agrees with the dynamical renormalization group derived by directly coarse graining the equations of motion.

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I. INTRODUCTION

The goal of this paper is to investigate the renormalization group (RG) approach to nonequilibrium field theory. We derive the renormalization group from iterative coarse graining of the Schwinger-Keldysh or closed-time-path (CTP) action [1,2]. We work out a specific example where variation of the CTP action leads to the so-called Kardar-Parisi-Zhang (KPZ) equation [3–5]. We show that the renormalization group obtained from the coarse grained action (CGA), agrees with the dynamical RG derived by directly coarse graining the equations of motion [3,4].

The RG [6–8] is a powerful method by which to analyze complex physical systems. Given a description of the system at some scale, a new description at a lower level of resolution is derived by coarse graining the former. By analyzing how the picture of the system changes (or fails to change) with resolution, important physical information is derived.

A field theory is most often not considered a fundamental description of a physical system. Its field variables are considered as the relevant degrees of freedom at some degree of resolution. This description is not complete, leaving out some uncontrolled sector whose interaction with the field variables is characterized as noise and dissipation [9,10]. We would like to associate with each level of description a corresponding action, so that the changes in this action as we change the resolution of our description allow us to define the dynamical RG for the theory.

A simple way of implementing this idea is by looking at the CTP generating functional, whose Legendre transform yields the CTP effective action (EA). The generating functional admits a representation as a path integral over fluctuations in the field variables of the exponential of an action functional. By performing a partial integration over some fluctuations, we obtain a new integrand which may be used to define the CGA [11]. The change in the CGA as more fluctuations are integrated away defines the dynamical RG.

In equilibrium, there is an efficient way to code a description of the system through some adequate thermodynamic potential (the free energy for a system in canonical equilibrium, etc.). In field theory, the proper thermodynamic potential under canonical equilibrium conditions is the Euclidean action, where the time variable is identified with periodicity $\beta=1/T$, and T is the temperature. The Euclidean CGA is defined from a partial integration over the field variables; the variation of the Euclidean CGA with scale is given by the Wegner-Houghton equation [12], and gives rise to the so-called exact RG [13].

In dynamical situations, such devices are not forthcoming, and so the dynamical RG is usually formulated at the level of the equations of motion [6]. Since thermodynamic potentials are most often simpler than equations of motion, the equilibrium RG has been much better developed than the dynamical RG.

In dynamical situations, the Lorentzian EA, which may be used, for example, to derive *S*-matrix elements of the field operators, cannot be used to derive a physically sound evolution for the background fields [14]. A simple solution lies in adopting the so-called Schwinger-Keldysh techniques [15,16]. In this paper, we show that essentially the same ideas can be used to define a convenient CTP action for stochastic field theories [17]. For the heat diffusion equation near equilibrium this was done in Ref. [18].

The basic element of the Schwinger-Keldysh or CTP method is the doubling of degrees of freedom. For each field variable in the original theory, a new mirror variable is introduced; accordingly, the number of external sources in the generating functional is also doubled, and the EA is defined as a Legendre transform with respect to *all* variables independently. The dynamics for the background (also called classical or mean) fields is obtained by taking the variation of the EA, and then (but only then) imposing some constraint on the mirror variables, in order to eliminate the excess degrees of freedom. The formalism is built in such a way as to make sure that the resulting dynamics is causal and respects the reality of the background fields.

We wish to point out that there are other implementations of the doubling of degrees of freedom idea. The best known in this context is possibly the so-called Martin-Rose-Siggia

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formalism [19] for stochastic differential equations (SDE), which is closely related to the CTP approach [20].

Since the CTP EA may be used to derive real and causal equations of motion for the expectation values of field operators (and, in an extension of the formalism, also for their correlations [10,21]) it is natural to define the RG for nonequilibrium field theory from the iterated coarse graining of the CTP action. This approach to the RG has been put forward in Refs. [1] and [2]. These authors show that, under the adiabatic approximation, the RG defined from the CTP action reduces to the usual (equilibrium) one.

However, nonequilibrium field theories also manifest a regime (called strongly dissipative by Berera *et al.* [22]) with very different properties from equilibrium fields. At the level of the CTP EA [23,24], this regime is characterized by the EA becoming complex, and also by the entanglement of the original field variables and their doubles, in such a way that the CTP EA no longer may be written as the difference of two independent action functionals.

These nonseparable terms are associated with dissipation (when they are real) and noise (when they are imaginary). The joint presence of noise and dissipation, which is due to the unitarity of the underlying theory, is the dynamical foundation of the fluctuation-dissipation relation near equilibrium.

On closer examination, it is not surprising that studies of the nonequilibrium RG in field theory so far have found no evidence of this strongly dissipative regime. The case is analogous to, for example, the situation in thermal field theory in four space-time dimensions. An approach to the renormalization group based on the ultraviolet behavior of correlation functions (which is insensitive to temperature [25]) will fail to disclose the nontrivial fixed point at very high temperatures, when the theory becomes effectively three dimensional. In the same way, the RG derived under the adiabatic approximation is insensitive to noise and dissipation, because these are nonadiabatic effects [9].

In the thermal case, what is needed is an ‘‘environmentally friendly’’ approach to the RG [26], where temperature dependent correlations and coupling constants are used throughout. In the nonequilibrium case, the starting point must be a noisy and dissipative CTP action, including new parameters associated with the nonadiabatic terms.

The stumbling block in the completion of this program is the lack of an efficient parametrization of the nonadiabatic CTP action. Of course, it is possible to start from the adiabatic action (that is, the difference of two Lorentzian actions for the field and mirror variables, respectively) and derive the nonadiabatic action after coarse graining some of the quantum fluctuations. However, the fact that this is necessarily done in some kind of perturbative scheme (which assumes that the resulting corrections to the action are small) to a large extent defeats the purpose of the whole exercise. However, the strongly dissipative regime of nonequilibrium field theory truly exists, to such extent that most practical applications of nonequilibrium field theory are actually based in stochastic classical field theory, kinetic theory, and even hydrodynamics, all of them limiting cases of the strongly dissipative regime. An interesting example is the slow rolling assumption made in inflationary cosmology (see, for example, Ref. [27]). There, the second time derivative of the inflationary field, which obeys a Klein-Gordon equation, is discarded when compared with the dissipative term.

The goal of this paper is to put forward the essential elements of the CTP approach to stochastic dynamics, and the derivation of the dynamical RG therefrom. Rather than an abstract presentation, we have chosen to work on a specific example. We have chosen a parametrization of the CTP action for a scalar field theory whose variation leads to the noisy KPZ equation in $3+1$ dimensions [3–5]. We have chosen this example because it is relatively simple, while its manifold applications warrant its physical cogency [5,28–30]. Related with the KPZ equation is the Burger’s equation [31] which, among its multiple applications, has been useful in describing problems of structure formation in cosmology [32].

The paper is organized as follows. In the following section, we introduce the basic notions regarding the CTP formalism; then we proceed to defining the CTP action and the CTP generating functional for SDE, and show the connection with the usual (single-path) functional formulation for SDE. We then apply the CTP approach to the KPZ equation, calculating its associated CTP EA. In Sec. III we introduce the CG generating functionals and the corresponding actions or CGA. In Sec. IV we study the way the CGA runs with changing coarse graining, and compare the resulting RG with the one derived by other means. We show that the resulting RG displays nontrivial running for the noise and dissipation terms in the action. We conclude with some brief final remarks in Sec. VI.

In Appendix A it is shown explicitly that the field equations derived from the CTP EA for the KPZ equation reproduce the right dynamics for the classical (i.e., mean or background) field. In Appendix B we compute the CGA to second order in the nonlinearity. In Appendix C we show that the effective theory for the modes that survive the coarse graining of the KPZ CTP generating functional, is equivalent to that obtained by coarse graining the equations of motion of the field.

II. CLOSED-TIME-PATH AND STOCHASTIC DIFFERENTIAL EQUATIONS

After a brief review of CTP formalism, we shall proceed to define the CTP action and the CTP generating functional for a class of SDE. Next, we shall apply our method to the specific example of the KPZ equation, computing the CTP EA and deriving from it the equation of motion for the classical field.

A. Closed-time-path field theory

In the usual In-Out formulation of the quantum field theory, the basic object is the vacuum persistence amplitude Z, which encodes all the dynamical information of the theory [33]. Suppose we are dealing with a scalar field theory. Then we define

$$Z[J] \equiv \langle \mathrm{Out}|\mathrm{In}\rangle_J = \int \mathcal{D}\Phi(x)\exp\left(iS[\Phi]+i\int J\Phi\right). \quad (1)$$

Here S is the action of the field Φ, and J is an external current coupled linearly to the field. (If not indicate explicitly, the integrals are over the entire space-time.) This functional generates matrix elements of operators between In and Out states, rather than proper expectation values referred to a single state. Hence, this formulation is useful when one asks questions about scattering problems or rate transitions, for instance. But if we want to deal, in this formalism, with the time evolution of true expectation values, we must be able to relate two different complete set of states, e.g., via Bogolubov coefficients, in order to relate the In and Out bases. Instead, we can use the functional integral method developed by Schwinger and Keldysh, known as CTP formalism [15]. In the In-Out formulation, when working in the Heisenberg picture, the vacuum persistence amplitude (1) is also given by

$$Z[J] = \left\langle \mathrm{Out}\left| T\exp\left[i\int d^4x\, J(x)\Phi_H(x)\right]\right|\mathrm{In}\right\rangle, \quad (2)$$

where T denotes temporal order, and Φ_H is the field operator in the Heisenberg picture. By contrast, in the CTP formulation we define a more symmetric object, namely,

$$\begin{aligned} Z[J^+,J^-] &= {}_{J^-}\langle \mathrm{In}|\mathrm{In}\rangle_{J^+} \\ &= \left\langle \mathrm{In}\left|\tilde{T}\exp\left[-i\int_{-\infty}^{t^*}dt\int d^3x\, J^-(x)\Phi_H(x)\right]\right.\right. \\ &\quad \left.\left.\times T\exp\left[i\int_{-\infty}^{t^*}dt\int d^3x\, J^+(x)\Phi_H(x)\right]\right|\mathrm{In}\right\rangle. \end{aligned} \quad (3)$$

That is, one compares the final states that result from the evolution of the In state under the influence of two external currents, J^+ and J^-. Here $\tilde{T}$ means antitemporal order, and t^* is some late time, which in practice it is chosen to be $+\infty$. It is easily seen that the derivatives of $Z[J^+,J^-]$ evaluated at $J^+=J^-$ generate true expectation values of product of fields. In terms of path integrals, $Z[J^+,J^-]$ has the following representation

$$\begin{aligned} Z[J^+,J^-] = \int_{\Phi^+(t^*)=\Phi^-(t^*)} &\mathcal{D}\Phi^+(x)\mathcal{D}\Phi^-(x') \\ &\times\exp\left[i\left(S[\Phi^+]-S[\Phi^-]\right.\right. \\ &\left.\left.+\int J^+\Phi^+ - \int J^-\Phi^-\right)\right]. \end{aligned} \quad (4)$$

The quantity $S[\Phi^+]-S[\Phi^-]$ is the CTP action or S_{CTP}. In Eq. (4) we integrate over histories Φ^+ and Φ^- that join at time t^*. As in the In-Out formalism, the classical equations of motion are obtained from the variation of the action with respect to the fields.

We can define a generating functional

$$W[J^+,J^-] = -i\ln Z[J^+,J^-], \quad (5)$$

and classical fields

$$\Phi^{\pm}_{cl}(x) = \pm\frac{\delta W[J^+,J^-]}{\delta J^{\pm}(x)}. \quad (6)$$

Next, we define the CTP EA as the Legendre transform of W, that is,

$$\Gamma[\Phi^+_{cl},\Phi^-_{cl}] = W[J^+,J^-] - \int J^+\Phi^+_{cl} + \int J^-\Phi^-_{cl}, \quad (7)$$

where it is understood that the currents have been expressed as functions of the classical fields, via relations (6). The equations of motion for the classical fields can be written as follows:

$$\frac{\delta\Gamma[\Phi^+_{cl},\Phi^-_{cl}]}{\delta\Phi^{\pm}_{cl}(x)} = \mp J^{\pm}(x). \quad (8)$$

The common value of Φ^+_{cl} and Φ^-_{cl}, when $J^+=J^-$, is real, because it is a true expectation value. It can be shown from the definitions above that it obeys a causal equation of motion (see, for example, Ref. [16]). On the contrary, in the In-Out formalism the classical field is not necessarily real, nor the equation of motion that it follows is causal, as it is not a proper expectation value, unless the In and Out states coincide.

It will be convenient to rephrase the CTP formalism in terms of $\phi=\Phi^+-\Phi^-$ and $\varphi=\Phi^++\Phi^-$. The CTP condition will be given by $\phi(t^*)=0$, and the classical equations of motion by

$$\frac{\delta S_{CTP}[\phi,\varphi]}{\delta\phi(x)} = -j(x), \quad (9)$$

$$\frac{\delta S_{CTP}[\phi,\varphi]}{\delta\varphi(x)} = -\mathbf{J}(x), \quad (10)$$

where

$$\mathbf{J} = \frac{J^+-J^-}{2}, \quad j = \frac{J^++J^-}{2}. \quad (11)$$

Moreover, we shall have

$$\phi_{cl}(x) = \frac{\delta W[\mathbf{J},j]}{\delta j(x)}, \quad \varphi_{cl}(x) = \frac{\delta W[\mathbf{J},j]}{\delta\mathbf{J}(x)}. \quad (12)$$

We obtain more symmetrical equations of motion

$$\frac{\delta\Gamma}{\delta\phi_{cl}(x)}[\phi_{cl},\varphi_{cl}] = -j, \quad (13)$$

$$\frac{\delta\Gamma}{\delta\varphi_{cl}(x)}[\phi_{cl},\varphi_{cl}] = -\mathbf{J}. \quad (14)$$

When $\mathbf{J}=0$, the first of these equations gives the physical equation of motion, and the second one is trivially satisfied.

B. CTP approach to stochastic differential equations

The causal and real evolution obtained from the CTP formalism, suggests that a CTP action that reproduces the Langevin equation for a stochastic theory described by a field φ, could as well be used to compute the correlations of the field and to derive the equation of motion for the classical (i.e., mean) field by employing the corresponding CTP EA. In its turn, the defined CTP generating functional can be used to implement the RG in the same fashion as it is implemented at the level of a thermodynamic partition function for systems in equilibrium. In this way, one would find an alternative route for the usual dynamical RG [1,2].

Let us consider a stochastic differential equation of the general form

$$\mathcal{T}[\varphi](x)\equiv\frac{\partial\varphi(x)}{\partial t}-K[\varphi](x)=\eta(x), \tag{15}$$

where $K[\varphi]$ represents a differential operator not including time derivatives, and where η is a zero mean stochastic function with Gaussian probability distribution. It can be seen that this equation is obtained from the following CTP action, as explained below:

$$S_{CTP}[\phi,\varphi]=c\int dx\,\phi(x)\mathcal{T}[\varphi](x)+c^2\frac{i}{2}\int dx\,dx'\,\phi(x)N(x,x')\phi(x'). \tag{16}$$

Here $N(x,x')$ is the two point correlation function of the noise η, and c is a dimensional constant that makes the action dimensionless. The dimensions of c will depend on the physical interpretation we give to φ, e.g., the potential in fluid mechanics, the height function in surface growth. The CTP generating functional for this action is

$$Z_{CTP}[\mathbf{J},j]=\int\mathcal{D}\phi\,\mathcal{D}\varphi\,\exp\left\{iS_{CTP}[\phi,\varphi]+i\int dx[\mathbf{J}(x)\varphi(x)+j(x)\phi(x)]\right\}. \tag{17}$$

[The CTP condition, $\phi(t\to\infty)=0$, is understood.] To arrive at Eq. (15) we observe that the term $\exp\{-(c^2/2)\int dx dx'\,\phi(x)N(x,x')\phi(x')\}$ in Eq. (17), can be written as the functional Fourier transform of an auxiliary functional. We find, up to a constant factor,

$$\exp\left\{-\frac{c^2}{2}\int dxdx'\,\phi(x)N(x,x')\phi(x')\right\}=\int\mathcal{D}\eta\,P[\eta]\exp\left(-ic\int dx\,\eta(x)\phi(x)\right), \tag{18}$$

where

$$P[\eta]=\exp\left\{-\frac{1}{2}\int dx\,dx'\,\eta(x)N^{-1}(x,x')\eta(x')\right\} \tag{19}$$

is the probability distribution of the noise η, and N^{-1} means the inverse matrix of N. Hence, we can write

$$Z_{CTP}[\mathbf{J},j]=\int\mathcal{D}\eta\,\mathcal{D}\phi\,\mathcal{D}\varphi\,P[\eta]\times\exp\left\{iS_{CTP}[\phi,\varphi,\eta]+i\int dx[\mathbf{J}(x)\varphi(x)+j(x)\phi(x)]\right\}, \tag{20}$$

where

$$S_{CTP}[\phi,\varphi,\eta]=c\int\{\phi(x)\mathcal{T}[\varphi](x)-\eta(x)\phi(x)\}dx. \tag{21}$$

The variation of this action with respect to ϕ, leads to Eq. (15). This method allows us to relate the imaginary part of the CTP action, quadratic in the field ϕ, to stochastic sources [34,2,9,35,36,24]. The constant c can be absorbed by redefining ϕ as $c^{-1}\phi$; therefore ϕ and φ will not have, in general, the same dimensions. The variation of Eq. (21) with respect to the field φ, gives the equation of motion for ϕ. This equation contains information about the evolution of the response functions for the physical field φ. This will be clear in Sec. III, where we use the equation for ϕ in deriving the Feynman rules for a particular example.

Let us show that the CTP generating functional (17) is, up to a Jacobian factor, the generating functional usually defined in the theory of SDE from a probabilistic approach [17,37–39], namely,

$$Z[J]=\int\mathcal{D}\eta\,P[\eta]\exp\left(i\int dx\,J(x)\varphi_s(x;\eta]\right). \tag{22}$$

Here $\varphi_s(x;\eta]$ is the solution—assumed unique—of Eq. (15) for a particular realization of the noise η, whose probability distribution is P (19). The derivatives of Z with respect to the external current J give the correlation functions of the field, where the average process is referred to the noise probability distribution. This formulation of the stochastic problem is equivalent to the Martin-Siggia-Rose formalism [19] (see Ref. [40]). We now demonstrate that it is also equivalent to a CTP formulation based on Eq. (17). Inserting the following identity in Eq. (22):

$$\int\mathcal{D}\varphi\,\delta[\varphi(x)-\varphi_s(x;\eta]]=1, \tag{23}$$

we obtain

$$Z[J]=\int \mathcal{D}\eta\,\mathcal{D}\varphi\, P[\eta]\delta\,[\varphi(x)-\varphi_s(x;\eta)]\times\exp\left\{i\int dx\, J(x)\varphi(x)\right\}. \tag{24}$$

Changing variables in the argument of the delta functional yields

$$Z[J]=\int \mathcal{D}\eta\,\mathcal{D}\varphi\, P[\eta]\delta\,[\mathcal{T}[\varphi](x)-\eta(x)]\times\mathcal{J}\exp\left\{i\int dx\, J(x)\varphi(x)\right\}, \tag{25}$$

where

$$\mathcal{J}=\det\left\{\frac{\delta\mathcal{T}[\varphi]}{\delta\varphi}\right\}=\det\left\{\frac{\partial}{\partial t}-\frac{\delta K[\varphi]}{\delta\varphi}\right\}$$

is the Jacobian associated with the change of variables in the delta functional. If the operator K does not contain any time derivative, the Jacobian, up to a field independent factor, is given by [17,40]

$$\mathcal{J}=\exp\left\{-\frac{1}{2}\int dx\,\frac{\delta K[\varphi]}{\delta\varphi(x)}\right\}. \tag{26}$$

The next step is to expand the delta functional in Fourier components, that is,

$$\delta[\mathcal{T}[\varphi](x)-\eta(x)]=\int \mathcal{D}\phi\,\exp\left\{i\int dx\,\phi(x)[\mathcal{T}[\varphi](x)-\eta(x)]\right\}, \tag{27}$$

and then replace this expression in Eq. (25). The integral over the noise is done explicitly using Eq. (19), yielding

$$Z[J]=\int \mathcal{D}\phi\,\mathcal{D}\varphi\,\mathcal{J}\exp\left\{iS_{CTP}[\phi,\varphi]+i\int J(x)\varphi(x)\right\}, \tag{28}$$

where the action S_{CTP} is given by Eq. (16) after absorbing the constant c into ϕ.

We see that, when the Jacobian $\mathcal{J}$ is field independent, the expression (28) can be identified with the CTP generating functional we defined before motivated by more heuristic considerations. This happens for a broad class of SDE [37], including the KPZ [41] and Navier-Stokes equation [39].

It can be seen that Eqs. (9) and (10) are equivalent to the equations proposed for the physical and the nonphysical field operators, respectively, in the work of Martin, Siggia, and Rose [19]. To show this, we adopt the notation of that paper, thus

$$K[\varphi](x_1)\equiv\int dx_2 U_2(x_1,x_2)\varphi(x_2)+\int dx_2dx_3U_3(x_1,x_2,x_3)\varphi(x_2)\varphi(x_3), \tag{29}$$

and $U_1(x_1)\equiv\eta(x_1)$. Hence, remembering that the noise term in Eq. (16) is recovered after Fourier transform the quadratic term in ϕ, the classical and physical (i.e., $\mathbf{J}=0$) equations of motion derived from the CTP action—Eqs. (9) and (10), will be

$$\frac{\delta S_{CTP}}{\delta\phi(x_1)}=\frac{\partial\varphi(x_1)}{\partial t}-\int dx_2U_2(x_1,x_2)\varphi(x_2)-\int dx_2dx_3U_3(x_1,x_2,x_3)\varphi(x_2)\varphi(x_3)=U_1(x_1), \tag{30}$$

$$\frac{\delta S_{CTP}}{\delta\varphi(x_1)}=-\frac{\partial\phi(x_1)}{\partial t}-\int dx_2\phi(x_2)U_2(x_2,x_1)-2\int dx_2dx_3U_3(x_2,x_3,x_1)\phi(x_2)\varphi(x_3)=0. \tag{31}$$

These are the same as Eqs. (2.1) and (3.1b) in Ref. [19].

To conclude this section we compare the CTP approach with the single-path approach to SDE of Ref. [17] (see also Refs. [37,38,42]). The difference arises in Eq. (25). If the integral over the noise is performed explicitly with the aid of the delta functional, we get

$$Z[J]=\int \mathcal{D}\varphi P[\mathcal{T}[\varphi]]\mathcal{J}\exp\left\{i\int dx\,J(x)\varphi(x)\right\}. \tag{32}$$

Using the definition (19), and leaving apart the Jacobian $\mathcal{J}$, we obtain the following single-path action:

$$S_{SP}=-\frac{1}{2}\int dx\,dx'\,\mathcal{T}[\varphi](x)N^{-1}(x,x')\mathcal{T}[\varphi](x'). \tag{33}$$

While it is a valid representation of the generating functional, this action cannot be used to generate the dynamics of the classical fields. To see this, suppose that the operator K in Eq. (15) is linear in φ, and moreover that the Green function G, associate with that equation, is causal. The assumption regarding the linearity of K implies that $S_{SP}[\varphi]$ is quadratic, and that the Jacobian $\mathcal{J}$ is field independent, so it can be ignored. Because $\mathcal{T}$ is a linear operator, the classical field associated with S_{SP} will obey the classical equation of motion obtained from the variation of S_{SP}. So, when an external current J is coupled to the field we obtain

$$\varphi_{cl}(x)=\int dx_1dx_2dx_3G(x,x_1)N(x_1,x_2)G^{\dagger}(x_2,x_3)J(x_3). \tag{34}$$

Because of causality the Green function verifies $G(x,x_1) \propto \theta(t-t_1)$ and $G^{\dagger}(x_2,x_3) \propto \theta(t_3-t_2)$, and hence we cannot affirm that φ obeys a causal evolution. Therefore the physical meaning of φ_{cl} is limited, as for the classical fields in the In-Out formalism of quantum field theory. This problem is related to the fact that operator $\mathcal{T}$ appears twice in Eq. (33). As noted in Ref. [37], the set of solutions associated with the variation of S_{SP} in Eq. (33), includes not only the solutions of Eq. (15), which are causal, but a set of spurious solutions.

III. CTP APPROACH TO THE KPZ EQUATION

In this section we apply CTP methods to the KPZ equation [3],

$$\frac{\partial \varphi}{\partial t} - \nu \nabla^2 \varphi - \frac{\lambda}{2} (\boldsymbol{\nabla} \varphi)^2 = \eta, \tag{35}$$

where η is the noise term, assumed to have some particular Gaussian statistics. The KPZ is a well known equation that belongs to a general class of stochastic nonlinear differential equations of diffusive type, for which our method can be extended straightforwardly.

There is a large amount of literature regarding the KPZ equation (see, for example, Ref. [5] and references therein). We mention here only a few points concerning it. In the context of fluid dynamics, the KPZ equation is derived for the case of free-vorticity, null-pressure fluid, $-\boldsymbol{\nabla}\varphi$ being the velocity field. When used to describe some phenomena related to surface growth [3,5], the KPZ equation is also derived as one of the simplest nonlinear extension of the Edwards-Wilkinson equation, φ measuring surface height. In addition, the KPZ equation is closely related with flame-front propagation [28], dissipative transport [29], and polymer physics [30], to quote some. When derived from Navier-Stokes equation, it is seen that the nonlinear coupling must be $\lambda = 1$, which is not necessarily the case in treating, for example, surface growth. The noiseless KPZ equation is Galilean invariant, a property which, in the case of a fluid, is inherited from the Navier-Stokes equation, and that in the context of surface growth is related to the rotation symmetry of the coordinate system. If the noise is white, and translation invariant, this symmetry is preserved by the noisy KPZ equation as well. This fact implies a nonperturbative result concerning the running of the coupling λ when the RG is implemented [4,43], thus reducing the number of independent scaling exponents. As in the paper of Forster, Nelson, and Stephen for the case of Navier-Stokes equation [44] (see also Ref. [45]), one can derive the RG equations by coarse graining the equation of motion of the field φ. We want to apply ideas concerning the CTP formulation of quantum field theory in order to implement this RG transformation at the level of a CTP generating functional.

We begin by defining the CTP action and generating functional for the KPZ equation. The free case is examined in order to implement the perturbative calculation of the EA for the interacting case.

A. CTP action and generating functional for KPZ equation

As demonstrated in Ref. [41] the Jacobian $\mathcal{J}$ associated with the KPZ equation is field independent, and hence we can define a CTP action for the KPZ equation which describes the stochastic dynamics of the field φ. In $3+1$ dimensions, absorbing c into ϕ Eq. (16) yields

$$S_{CTP}[\phi,\varphi] = \int d^4x \left\{ \phi(x) \mathcal{L} \varphi(x) - \frac{\lambda}{2} \phi(x) (\boldsymbol{\nabla}\varphi)^2(x) \right\} + \frac{i}{2} \int d^4x d^4x' \, \phi(x) N(x,x') \phi(x'), \tag{36}$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} - \nu \nabla^2. \tag{37}$$

$N(x,x')$ is the two point correlation function of the noise η, assumed Gaussian and having zero mean value. With these definitions and per the early discussion we see that the KPZ equation is attained.

B. The free case

If the nonlinearity is absent, i.e., if $\lambda = 0$, we are dealing with the free case, and the corresponding free action is

$$S_0[\phi,\varphi] = \int d^4x \; \phi(x) \mathcal{L} \varphi(x) + \frac{i}{2} \int d^4x d^4x' \, \phi(x) N(x,x') \phi(x'). \tag{38}$$

When linear couplings of the fields with external currents are included, the variation of the free action with respect to the fields, gives the classical equations of motion, namely,

$$\frac{\delta S_0}{\delta \phi(x)} = \mathcal{L} \varphi(x) + \int d^4x' N(x,x') \phi(x') = -j(x). \tag{39}$$

$$\frac{\delta S_0}{\delta \varphi(x)} = -\mathcal{L}^* \phi(x) = -\mathbf{J}(x), \tag{40}$$

where

$$\mathcal{L}^* = \frac{\partial}{\partial t} + \nu \nabla^2. \tag{41}$$

We can write the solutions to Eqs. (39) and (40) in terms of the fundamental solutions G and G^*, which satisfy

$$\mathcal{L}_x G(x,x') = \mathcal{L}_x^* G^*(x,x') = \delta^4(x-x'). \tag{42}$$

Explicitly (in $1+3$ dimensions) we have

$$G(x,x') = \frac{e^{-(\vec{x}-\vec{x}')^2/4\nu(t-t')}}{(4\pi\nu|t-t'|)^{3/2}} \theta(t-t'), \tag{43}$$

$$G^*(x,x')=-\frac{e^{(\vec{x}-\vec{x}')^2/4\nu(t-t')}}{(4\pi\nu|t-t'|)^{3/2}}\theta(t'-t). \tag{44}$$

Hence,

$$\phi(x)=\int d^4x_1 G^*(x,x_1)\mathbf{J}(x_1), \tag{45}$$

$$\varphi(x)=-\int d^4x_1 G(x,x_1)j(x_1) - i\int d^4x_1 d^4x_2 d^4x_3 G(x,x_1) \times N(x_1,x_2)G^*(x_2,x_3)\mathbf{J}(x_3). \tag{46}$$

(There is the freedom on adding some arbitrary solutions of the homogeneous equations. However, the only such solution which is bounded for all times is identically zero.)

If $\mathbf{J}=0$ then $\phi=0$, and φ is given by

$$\varphi(x)=-\int d^4x_1 \frac{e^{-(\vec{x}-\vec{x}_1)^2/4\nu(t-t_1)}}{(4\pi\nu|t-t_1|)^{3/2}}\theta(t-t_1)j(x_1). \tag{47}$$

This entails a causal evolution.

We define the generating functional $Z_0[\mathbf{J},j]$ for the free fields

$$Z_0[\mathbf{J},j]=\int \mathcal{D}\phi(x)\mathcal{D}\varphi(x)\exp\left(iS_0[\phi,\varphi]+i\int (j\phi+\mathbf{J}\varphi)\right). \tag{48}$$

The mean fields are obtained by differentiating $-i\ln Z_0$ with respect to the currents. Having in mind that for the free case the mean fields satisfy the same equations (39) and (40), we find (up to a normalization factor)

$$Z_0[\mathbf{J},j]=\exp\frac{i}{2}\int d^4x_1 d^4x_2\Big\{-\mathbf{J}(x_1)G(x_1,x_2)j(x_2) + j(x_1)G^*(x_1,x_2)\mathbf{J}(x_2)-i\mathbf{J}(x_1) \times\Big[\int d^4x_3 d^4x_4 G(x_1,x_3) \times N(x_3,x_4)G^*(x_4,x_2)\Big]\mathbf{J}(x_2)\Big\}. \tag{49}$$

The two point correlation functions are given by the second derivatives of Z_0, so we have

$$\langle\phi(x)\phi(x')\rangle=0,$$

$$\langle\varphi(x)\varphi(x')\rangle = -\int d^4x_1\int d^4x_2 G(x,x_1)N(x_1,x_2)G^*(x_2,x'), \tag{50}$$

$$\langle\phi(x)\varphi(x')\rangle=-iG^*(x,x').$$

The corresponding functions in the momentum space are

$$\langle\phi(p)\phi(p')\rangle=0,$$

$$\langle\varphi(p)\varphi(p')\rangle=\frac{N(p,p')}{[ip^0+\nu\vec{p}^2][ip'^0+\nu\vec{p}'^2]}, \tag{51}$$

$$\langle\phi(p)\varphi(p')\rangle=\frac{i\delta(p+p')}{[ip'^0+\nu\vec{p}'^2]}.$$

[We indicate the Fourier transformed fields with the same name as the original fields, and use the following convention in $d+1$ dimensions: $f(p)=(2\pi)^{-(d+1)/2}\int d^d\vec{x}dx^0\exp\{-i(p^0x^0-\vec{p}\cdot\vec{x})\}f(x)$, where $x^0=t$.]

C. The interacting case

Let us go back to the definition of the generating functional for interacting fields (dropping the CTP subscripts)

$$Z[\mathbf{J},j]=\int \mathcal{D}\phi\mathcal{D}\varphi\exp\Big(iS[\phi,\varphi] + i\int d^4x[j(x)\phi(x)+\mathbf{J}(x)\varphi(x)]\Big). \tag{52}$$

The EA is given by

$$\Gamma[\phi_{cl},\varphi_{cl}]=-i\ln Z-\int d^4x[\phi_{cl}(x)j(x)+\varphi_{cl}(x)\mathbf{J}(x)]. \tag{53}$$

and admits the CTP representation

$$e^{i\Gamma[\phi_{cl},\varphi_{cl}]}=\int_{1\mathrm{PI}}\mathcal{D}\phi(x)\mathcal{D}\varphi(x)e^{iS[\phi_{cl}+\phi,\varphi_{cl}+\varphi]}, \tag{54}$$

where 1PI indicates that only diagrams one particle irreducible must be included in the diagrammatic evaluation of the functional integrals.

For the KPZ CTP action (36) we have

$$S[\phi_{cl}+\phi,\varphi_{cl}+\varphi]=S[\phi_{cl},\varphi_{cl}]+S_0[\phi,\varphi] -\frac{\lambda}{2}\int d^4x[\phi(\nabla\varphi)^2+\phi_{cl}(\nabla\varphi)^2 +2\phi\nabla\varphi\cdot\nabla\varphi_{cl}]_x + \text{linear terms in } \phi \text{ and } \varphi. \tag{55}$$

Taking the logarithm of Eq. (54) and expanding to $O(\lambda)^2$, it results

$$\Gamma[\phi_{cl},\varphi_{cl}]=S[\phi_{cl},\varphi_{cl}]-\frac{\lambda}{2}\int d^4x\langle[\phi(\nabla\varphi)^2+\phi_{cl}(\nabla\varphi)^2+2\nabla\varphi\cdot\nabla\varphi_{cl}]_x\rangle+\frac{i\lambda^2}{8}\int d^4x d^4x'\langle[\phi(\nabla\varphi)^2+\phi_{cl}(\nabla\varphi)^2+2\phi\nabla\varphi\cdot\nabla\varphi_{cl}]_x\times[\phi(\nabla\varphi)^2+\phi_{cl}(\nabla\varphi)^2+2\phi\nabla\varphi\cdot\nabla\varphi_{cl}]_{x'}\rangle_{connected}, \quad (56)$$

where the averaging operation $\langle\cdots\rangle$ is defined as

$$\langle\mathcal{F}[\phi,\varphi]\rangle\equiv\frac{\int\mathcal{D}\phi\mathcal{D}\varphi e^{iS_0[\phi,\varphi]}\mathcal{F}[\phi,\varphi]}{\int\mathcal{D}\phi\mathcal{D}\varphi e^{iS_0[\phi,\varphi]}}. \quad (57)$$

Note that the term $\langle(\phi\nabla\varphi\cdot\nabla\varphi_{cl})_x(\phi\nabla\varphi\cdot\nabla\varphi_{cl})_{x'}\rangle$, which could give a nontrivial equation of motion for ϕ_{cl}, vanishes, because it is proportional (up to spatial derivatives) to

$$\langle\phi(x)\varphi(x')\rangle\langle\phi(x')\varphi(x)\rangle\propto\theta(t'-t)\theta(t-t').$$

Since S_0 is quadratic, expectation values may be written as products of the two-point correlation functions given in Eq. (50) and (51). Hence, after Fourier transformation of Eq. (56), the result is

$$\Gamma[\phi_{cl},\varphi_{cl}]=S[\phi_{cl},\varphi_{cl}]+\Delta S[\phi_{cl},\varphi_{cl}], \quad (58)$$

where

$$\Delta S[\phi_{cl},\varphi_{cl}]=\frac{\lambda}{8\pi^2}\int d^4p_1d^4p_2\Delta_{ii}(p_1-p_2,p_2)\phi_{cl}(-p_1)+\frac{i\lambda^2}{64\pi^2}\Bigg\{\int d^4p_1d^4p_2\frac{-\Delta_{ij}(p_2,-p_2)(\vec p_1)_i(\vec p_1+\vec p_2)_j}{[-p_1^0+\nu\vec p_1^{\,2}][-(p_1+p_2)^0+\nu(\vec p_1+\vec p_2)^2]}$$
$$+4i\int d^4p_1d^4p_2d^4p_3\frac{\Delta_{ij}(p_2,p_3)(\vec p_2-\vec p_1)_i(-\vec p_1+\vec p_2+\vec p_3)_j}{[(p_1-p_2)^0+\nu(\vec p_1-\vec p_2)^2]}\phi_{cl}(-p_1)\varphi_{cl}(p_1-p_2-p_3)$$
$$+\int d^4p_1d^4p_2d^4p_3d^4p_4\Delta_{ij}(p_2,p_3)\Delta_{ij}(p_1-p_2,p_3-p_4)\phi_{cl}(-p_1)\phi_{cl}(-p_3)\Bigg\}. \quad (59)$$

The sum over repeated indices is understood, and

$$\Delta_{ij}(p,p')=\frac{N(p,p')}{[ip^0+\nu\vec p^{\,2}][ip'^0+\nu\vec p'^{\,2}]}p_ip'_j. \quad (60)$$

The equations of motion for the classical fields result from the first variations of the EA. Those with proper physical meaning are obtained when ϕ, $\mathbf{J}=0$. Thus, to $O(\lambda^2)$,

$$\frac{\delta\Gamma}{\delta\phi_{cl}(-p)}[\phi_{cl}=0,\varphi_{cl}]=[ip^0+\nu\vec p^{\,2}]\varphi_{cl}+\frac{\lambda}{8\pi^2}\int d^4p_1\vec p_1\cdot(\vec p-\vec p_1)\varphi_{cl}(p_1)\varphi_{cl}(p-p_1)+\frac{\lambda}{8\pi^2}\int d^4p_1\Delta_{ii}(p-p_1,p_1)$$
$$-\frac{\lambda^2}{16\pi^4}\int d^4p_1d^4p_2\frac{\Delta_{ij}(p_1,p_2)(\vec p_1-\vec p)_i(\vec p_1+\vec p_2-\vec p)_j}{[(p-p_1)^0+\nu(\vec p-\vec p_1)^2]}\varphi_{cl}(p-p_1-p_2)=-j(p). \quad (61)$$

The equation

$$\frac{\delta\Gamma}{\delta\varphi_{cl}(-p)}[\phi_{cl}=0,\varphi_{cl}]=0 \quad (62)$$

is automatically satisfied.

In Appendix A we show explicitly that Eq. (61) is also obtained by averaging the KPZ equation (36).

IV. COARSE GRAINED CTP ACTION FOR THE KPZ EQUATION

In order to implement the RG transform, we analyze the influence that the modes of higher wave number exert on lower ones, by computing the CGA [1,2,11,46]. When we are only concerned with the lower wave number sector of the theory, we can carry out explicitly, in the generating functional Z, the integration over the higher wave number modes, and the result of this partial integration will be a functional of the lower wave number modes only. This functional is indeed a generating functional for the lower modes, in which the influence of the higher modes is incorporated as modifications of the original action.

This procedure may be seen as a straightforward application of the Feynman-Vernon influence functional techniques to this problem [34], where the low wave number sector is regarded as "system" and the short wave modes as "bath."

To the best of our knowledge this approach has not been systematically discussed in the literature on SDE. There exist works where the RG is also derived from functional formulations of the stochastic theory (see, for example, Ref. [40] for critical dynamics of helium, antiferromagnetics, and liquid-gas systems; or Refs. [43,47] for the KPZ equation). The crucial difference between these works and the present paper is that we coarse grain the generating functional explicitly, imposing an ultraviolet cutoff, while in the cited papers the RG is obtained from the study of the singular ultraviolet contributions to the many-points response functions.

We start with our early definition (52) of the generating functional for the interacting fields

$$Z[\mathbf{J},j]=\int \mathcal{D}\phi\mathcal{D}\varphi\exp\bigg(iS[\phi,\varphi] + i\int d^4x[j(x)\phi(x)+\mathbf{J}(x)\varphi(x)]\bigg). \tag{63}$$

Now we split up the field and currents, according a scale that we shall choose below,

$$\varphi=\varphi_{>}+\varphi_{<},$$
$$\phi=\phi_{>}+\phi_{<},$$
$$j=j_{>}+j_{<},$$
$$\mathbf{J}=\mathbf{J}_{>}+\mathbf{J}_{<}.$$

Here, $\varphi_{>}$ contains the modes of higher wave number, $\varphi_{<}$ contains the lower ones, and analogously for the other quantities. The division will be specified by a cutoff Λ_s,

$$\varphi_{<}(x)=\int_{|\vec{k}|<\Lambda_s}\frac{d^4k}{(2\pi)^2}e^{i(k^0x^0-\vec{k}\cdot\vec{x})}\varphi(k^0,\vec{k}), \tag{64}$$

$$\varphi_{>}(x)=\int_{\Lambda_s<|\vec{k}|<\Lambda}\frac{d^4k}{(2\pi)^2}e^{i(k^0x^0-\vec{k}\cdot\vec{x})}\varphi(k^0,\vec{k}), \tag{65}$$

and so on. Here, $\varphi(k)$ is the Fourier transform of the field $\varphi(x)$, and Λ can be identified with a natural cutoff of the theory.

In any case, the correlations are obtained from the variation of Z with respect to the currents, and after that, by setting the currents equal to zero. As stated earlier, we just want to compute correlation functions involving the lower wave number modes. That can be accomplished merely by the variation of Z with respect to $j_{<}$ and $\mathbf{J}_{<}$. Therefore, it will be enough if we set, from the beginning, $j_{>},\mathbf{J}_{>}=0$. The CGA is achieved by performing explicitly the functional integrations over $\varphi_{>}$ and $\phi_{>}$. We rewrite the action $S[\phi,\varphi]=S[\phi_{>}+\phi_{<},\varphi_{>}+\varphi_{<}]$ in the following manageable way, which will be useful to compute the CGA perturbatively,

$$S[\phi_{>}+\phi_{<},\varphi_{>}+\varphi_{<}]=S[\phi_{<},\varphi_{<}]+S_0[\phi_{>},\varphi_{>}] + S_I[\phi_{>},\phi_{<},\varphi_{>},\varphi_{<}]. \tag{66}$$

Here, S_0 corresponds to the free action of the original theory. Hence, it results

$$\begin{aligned}Z[\mathbf{J}_{<},j_{<}]&=\int \mathcal{D}\varphi_{<}\mathcal{D}\phi_{<}\exp\bigg(iS[\phi_{<},\varphi_{<}] + i\int d^4x[j_{<}\phi_{<}+\mathbf{J}_{<}\varphi_{<}](x)\bigg)\\&\quad\times\bigg\{\int \mathcal{D}\varphi_{>}\mathcal{D}\phi_{>}e^{iS_0[\phi_{>},\varphi_{>}]+iS_I[\phi_{<},\phi_{>},\varphi_{>},\varphi_{<}]}\bigg\}\\&=\int \mathcal{D}\varphi_{<}\mathcal{D}\phi_{<}\exp\bigg(iS[\phi_{<},\varphi_{<}] + i\int d^4x[j_{<}\phi_{<}+\mathbf{J}_{<}\varphi_{<}](x)\bigg)e^{i\Delta S[\phi_{<},\varphi_{<}]}.\end{aligned} \tag{67}$$

The CGA is defined as

$$S_{CG}[\phi_{<},\varphi_{<}]=S[\phi_{<},\varphi_{<}]+\Delta S[\phi_{<},\varphi_{<}]. \tag{68}$$

In the present paper we are concerned with the KPZ equation and with its associated CTP action (36), which in p space is given by

$$\begin{aligned}S[\phi,\varphi]&=\int d^4p\ \phi(-p)(ip^0+\nu\vec{p}^2)\varphi(p)\\&\quad+\frac{\lambda}{2}\int d^4p_1d^4p_2d^4p_3(2\pi)^{-2}\delta(p_1+p_2+p_3)\\&\quad\times\vec{p}_2\cdot\vec{p}_3\phi(p_1)\varphi(p_2)\varphi(p_3)\\&\quad+\frac{i}{2}\int d^4p_1d^4p_2\phi(p_1)N(-p_1,-p_2)\phi(p_2).\end{aligned} \tag{69}$$

Splitting the fields according to wave number yields

$$S[\phi,\varphi]=\int d^4p\ \phi_>(-p)(ip^0+\nu\vec{p}^2)\varphi_>(p)+\int d^4p\ \phi_<(-p)(ip^0+\nu\vec{p}^2)\varphi_<(p)$$
$$+\frac{i}{2}\int d^4p_1d^4p_2\phi_>(p_1)N(-p_1,-p_2)\phi_>(p_2)+\frac{i}{2}\int d^4p_1d^4p_2\phi_<(p_1)N(-p_1,-p_2)\phi_<(p_2)$$
$$+i\int d^4p_1d^4p_2\phi_>(p_1)N(-p_1,-p_2)\phi_<(p_2)+\frac{\lambda}{2}\int d^4p_1d^4p_2d^4p_3(2\pi)^{-2}\delta(p_1+p_2+p_3)\vec{p}_2\cdot\vec{p}_3\{\phi_{>1}\varphi_{>2}\varphi_{>3}$$
$$+\phi_{<1}\varphi_{>2}\varphi_{>3}+2\phi_{<1}\varphi_{>2}\varphi_{<3}+2\phi_{>1}\varphi_{>2}\varphi_{<3}+\phi_{>1}\varphi_{<2}\varphi_{<3}+\phi_{<1}\varphi_{<2}\varphi_{<3}\}. \tag{70}$$

We shall assume that the noise is translation invariant (TI), therefore the term in the third line is zero because of orthogonality. Hence, as before, we have

$$S_0[\phi_>,\varphi_>]=\int d^4p\ \phi_>(-p)(ip^0+\nu\vec{p}^2)\varphi_>(p)+\frac{i}{2}\int d^4p_1d^4p_2\phi_>(p_1)N(-p_1,-p_2)\phi_>(p_2), \tag{71}$$

and using the definition given in Eq. (66), we find

$$S_I[\phi_>,\phi_<,\varphi_>,\varphi_<]=\frac{\lambda}{2}\int d^4p_1d^4p_2d^4p_3(2\pi)^{-2}\delta(p_1+p_2+p_3)\vec{p}_2\cdot\vec{p}_3\{\phi_{>1}\varphi_{>2}\varphi_{>3}+\phi_{<1}\varphi_{>2}\varphi_{>3}+2\phi_{<1}\varphi_{>2}\varphi_{<3}$$
$$+2\phi_{>1}\varphi_{>2}\varphi_{<3}+\phi_{>1}\varphi_{<2}\varphi_{<3}\}. \tag{72}$$

Therefore, from Eq. (67), we obtain

$$e^{i\Delta S[\phi_<,\varphi_<]}=\int \mathcal{D}\phi_>\mathcal{D}\varphi_>e^{iS_0[\phi_>,\varphi_>]}\exp\Bigg[\frac{i\lambda}{2}\int d^4p_1d^4p_2d^4p_3(2\pi)^{-2}\delta(p_1+p_2+p_3)\vec{p}_2\cdot\vec{p}_3\{\phi_{>1}\varphi_{>2}\varphi_{>3}+\phi_{<1}\varphi_{>2}\varphi_{>3}$$
$$+2\phi_{<1}\varphi_{>2}\varphi_{<3}+2\phi_{>1}\varphi_{>2}\varphi_{<3}+\phi_{>1}\varphi_{<2}\varphi_{<3}\}\Bigg]. \tag{73}$$

When the noise is white, TI, and has no spatial correlations, we have

$$N(p,p')=2D\ \delta(p^0+p'^0)\delta(\vec{p}+\vec{p}'), \tag{74}$$

where D is the noise amplitude. For this case (see details in Appendix B),

$$\Delta S[\phi_<,\varphi_<]=-\frac{\lambda}{2}\int d^4p(2\pi)^2\delta(p)\mathcal{F}\phi_<(p)+\frac{i}{2}\int d^4p\ \phi_<(-p)2\ \delta D\phi_<(p)$$
$$+i\lambda^2D\int d^4q\left(\int\frac{d^4p}{4\pi^2}\phi_<(-p)\varphi_<(p-q)A(p,q)\right)\left(\int\frac{d^4p}{4\pi^2}\phi_<(-p)\varphi_<(p+q)A(p,-q)\right)$$
$$+2D\lambda^2\int d^4p\ \phi_<(-p)\varphi_<(p)\times\vec{p}^2\delta\nu(p)-\frac{\lambda^2}{2}\int d^4p\ d^4q\ d^4k\ d^4l\ C(p,q,k,l)\phi_<(-p)\varphi_<(q)\varphi_<(k)\varphi_<(l). \tag{75}$$

We have defined the tadpole amplitude,

$$\mathcal{F}=\int\frac{d^4q}{16\pi^4}\frac{2D\vec{q}^2\mathbf{M}(q)}{[(q^0)^2+\nu^2(\vec{q}^2)^2]}, \tag{76}$$

the noise amplitude correction

$$2\delta D=2D^2\lambda^2\int\frac{d^4q}{4\pi^2}\times\frac{[\vec{q}\cdot(\vec{p}-\vec{q})]^2\mathbf{M}(q,p-q)}{[(q^0)^2+\nu^2(\vec{q}^2)^2][(p^0-q^0)^2+\nu^2\{(\vec{p}-\vec{q})^2\}^2]}, \tag{77}$$

the function A, related to the arising of multiplicative noise,

$$A(p,q)=\frac{(\vec{p}-\vec{q})\cdot\vec{q}}{(iq^0+\nu\vec{q}^2)}\mathbf{M}(q), \tag{78}$$

the viscosity correction

$$\vec{p}^2\delta\nu(p)=\int\frac{d^4q}{16\pi^4}\times\frac{\vec{q}\cdot(\vec{p}-\vec{q})\vec{p}\cdot(\vec{p}-\vec{q})\mathbf{M}(q,p-q)}{(iq^0+\nu\vec{q}^2)[(p^0-q^0)^2+\nu^2\{(\vec{p}-\vec{q})^2\}^2]}, \tag{79}$$

and the $\varphi^3_<$-interaction coupling

$$C(p,q,k,l)=(2\pi)^{-4}\delta(-p+q+k+l)\times\frac{\vec{q}\cdot(\vec{k}+\vec{l})\vec{k}\cdot\vec{l}\,\mathbf{M}(k+l)}{[i(k^0+l^0)+\nu(\vec{k}+\vec{l})^2]}. \tag{80}$$

In its turn, $M(p,q,\ldots,k)$ means that the momenta in the set $\{p,q,\ldots,k\}$ are restricted (i.e, must be projected) to the momentum shells $\Lambda_s<|\vec{p}|<\Lambda$, $\Lambda_s<|\vec{q}|<\Lambda$, and so on.

In conclusion, when Eq. (63) is coarse grained, the generating functional for the remaining modes is obtained by modifying the original viscosity ν and the noise amplitude D, and by adding some new terms: a tadpole term that concerns the homogenous mode only, a multiplicative noise term (see Appendix B), and a cubic interaction term. We remark that the noise terms are read directly from the imaginary part of the CTP CGA.

V. RENORMALIZATION GROUP FROM CTP CGA

The action S we started with (36), is actually a coarse grained action. The fields φ and ϕ are assumed to describe the physical world up to certain degree of resolution, limited, eventually, by a natural cutoff Λ, as can be the atomic size in a turbulent fluid or the Compton length of heavy particles in particle physics. When we integrated the higher wave number modes in the generating functional, we obtained a new action, suitable for a physical description with a lower degree of resolution Λ_s.

Suppose that we are interested in the behavior of the theory at momentum scales not superior than $e^{-s}\Lambda$, with s real and positive. In principle, this implies that, in the integrations we performed in Sec. IV, some linear combinations of the momenta must be restricted to the shell $e^{-s}\Lambda\leq|\vec{p}|\leq\Lambda$. Often we are only concerned with the small-p modes, for which p is near to 0. Hence, we must integrate all the modes except those very close to the origin, as close as necessary to obtain a leading order result. However, it can be seen that in the case of the KPZ equation, as in others, divergences arise in the limit of $\Lambda_s\to 0$, indicating that the perturbative approach fails [3–5].

What we can do instead, is to implement the so called RG formalism [6–8]. The scheme is the following.

(i) The integration from Λ to $\Lambda\to 0$ is performed not at once, but in repeated integrations over infinitesimal shells in three-momentum space. In integrating over one such shell, the cutoff changes from Λ' to $e^{-s}\Lambda'\approx(1-\delta s)\Lambda'$.

(ii) After each shell is integrated, the fields, lengths, times, momenta, etc., must be rescaled to bring the theory to its original aspect. In particular, the rescaling of momenta must adjust the cutoff to its initial value at the beginning of the process, and, in addition, some factors can affect the coupling constants.

When combined and repeated these operations give sensible results. It must be clear that we are not simply calculating an integral as the sum of discrete contributions from a partition of the domain, because at each step the coupling constant that measures the perturbation is renormalized, that is, at each step we are perturbating with respect to a different coupling constant; it is an iterative process that gives meaning to the whole integration between Λ and $\Lambda\to 0$.

In Sec. IV we have already performed the first step of the RG scheme: the coarse graining of the generating functional. The result was that the generating functional for the long modes is obtained from the original one by introducing (i) a correction to the noise correlation function, given in Eq. (77) and (ii) a correction to the viscositylike coupling, given in Eq. (79), and finally by including a set of new terms: the first (tadpole term), the third (multiplicative noise term), and the fifth (cubic interaction term) in Eq. (75). Extra terms are a common byproduct when one coarse grains a generating functional [6]. Actually, the tadpole term can be eliminated by a simple transformation, $\varphi_<(p)\to\varphi_<(p)+i2\pi^2\lambda\mathcal{F}\partial_0\delta(p)$; thus only the multiplicative noise (MN) and the cubic interaction (CI) terms remain. At this point, if we proceed further and repeat the coarse graining, it can be seen that, at $O(\lambda^2)$, no others terms arise. The effect of the MN and that of the CI is just to correct the terms already present in the CGA in a way that can be traced systematically. Hence, the first time we do the coarse graining is very special, because the effective viscosity is now a momentum dependent function, new terms arise that were not included in the original action, and no other terms appear when we repeat the coarse graining.

The natural question is why not to include this momentum dependence and the new terms from the very beginning. The momentum dependence of the viscosity can be ignored if one is interested in the $\vec{p}\to 0$ limit of the theory only. The MN and CI terms, because of the constraint that some momenta must be on the shell, involve modes that despite being $<$, must lie close to the shell. Hence, if it is assumed that the fields $<$ have support near the origin, these extra terms vanish. Moreover, under certain assumptions MN and CI terms, as more and more shells are integrated and the variables rescaled, tend to vanish, i.e., they are irrelevant terms. We shall not include them in our treatment of the RG (see below). In what follows we shall work out in an arbitrary number of spatial dimensions d, and, furthermore, assume that initially $\Lambda=1$, with the appropriate dimensions. As before, the noise verifies Eq. (74).

Let us study the small-p limit of the corrections introduced by the coarse graining. We start with Eq. (77). There,

q and $p-q$ must be on the shell between $(1-\delta s)\Lambda$ and Λ, and because we shall restrict ourselves to the small-p limit, we must inspect the behavior of the integral when the external momenta p get close to 0. To lowest order, the effective noise satisfies Eq. (74), provided D is adjusted by the following amount [3–5]:

$$\delta D=\frac{\lambda^2 D K_d}{4\nu^3}\delta s, \tag{81}$$

where $K_d=S_d/(2\pi)^d$, and S_d is the area of a unit sphere in d dimensions. A similar conclusion is reached for Eq. (79) [3–5]; in the small-p limit, we find that ν must be replaced by $\nu+\delta\nu$, where

$$\delta\nu=-K_d\frac{\lambda^2 D}{\nu^2}\frac{d-2}{4d}\delta s. \tag{82}$$

Hence, when attention is paid to the small-p modes, the CGA will be given by (eliminating the tadpole, discarding MN and CI terms, and dropping the subscripts $<$)

$$\begin{aligned}S_{CG}[\phi,\varphi]=&\int d^{d+1}p\ \phi(-p)(ip^0+[\nu+\delta\nu]\vec{p}^2)\varphi(p)\\&+\frac{\lambda}{8\pi^2}\int d^{d+1}p_1 d^{d+1}p_2 d^{d+1}p_3\delta(p_1+p_2+p_3)\\&\times\vec{p}_2\cdot\vec{p}_3\phi(p_1)\varphi(p_2)\varphi(p_3)\\&+\frac{i}{2}\int d^{d+1}p\ \phi(-p)[2D+2\delta D]\phi(p).\end{aligned} \tag{83}$$

Next, we proceed with the rescaling. We take $b=1+\delta s$, and define

$$\varphi(p^0,\vec{p})=b^{\alpha+z+d}\tilde{\varphi}(p'^0,\vec{p}'), \tag{84}$$

$$\phi(p^0,\vec{p})=b^{-\alpha+z}\tilde{\phi}(p'^0,\vec{p}'), \tag{85}$$

where $p'^0=b^z p^0$ and $\vec{p}'=b\,\vec{p}$. Therefore, we obtain

$$\begin{aligned}S_{CG}[\phi,\varphi]&=\tilde{S}[\tilde{\phi},\tilde{\varphi}]\\&=\int d^{d+1}p\ \tilde{\phi}(p)(ip^0+\{b^{z-2}[\nu+\delta\nu]\}\vec{p}^2)\tilde{\varphi}(p)\\&\quad+b^{\alpha+z-2}\frac{\lambda}{8\pi^2}\int d^{d+1}p_1 d^{d+1}p_2 d^{d+1}p_3\\&\quad\times\delta(p_1+p_2+p_3)\vec{p}_2\cdot\vec{p}_3\tilde{\phi}(p_1)\tilde{\varphi}(p_2)\tilde{\varphi}(p_3)\\&\quad+\frac{i}{2}\int d^{d+1}p\ \tilde{\phi}(-p)\{b^{-2\alpha-d+z}\\&\quad\times[2D+2\delta D]\}\tilde{\phi}(p).\end{aligned} \tag{86}$$

Some remarks are in order. The new variable p is such that $|\vec{p}|$ runs up to Λ, as for the original fields. The exponent $\alpha+z+d$, which rescales the field $\varphi(p)$ in the momentum representation, matches with an exponent equal to α for the rescaling of $\varphi(x)$. The choice of the exponent given for ϕ, makes the free part of the rescaled CGA form invariant, and hence, we can iterate the process without further modifications.

In conclusion, after integration and rescaling are performed, the action is characterized by a different viscosity,

$$\tilde{\nu}=(1+\delta s)^{z-2}[\nu+\delta\nu], \tag{87}$$

by a new coupling constant

$$\tilde{\lambda}=(1+\delta s)^{\alpha+z-2}\lambda, \tag{88}$$

and by a new noise correlation coefficient

$$\tilde{D}=(1+\delta s)^{-2\alpha-d+z}[D+\delta D]. \tag{89}$$

These are general relations, which are valid for every two consecutive instances of the RG procedure. Finally, we can arrive at a set of differential equations for the running of these quantities, namely,

$$\frac{d\nu}{ds}=\nu\left[z-2-K_d\frac{\lambda^2 D}{\nu^3}\frac{d-2}{4d}\right], \tag{90}$$

$$\frac{d\lambda}{ds}=\lambda[\alpha+z-2], \tag{91}$$

$$\frac{dD}{ds}=D\left[z-d-2\alpha+\frac{\lambda^2 D K_d}{4\nu^3}\right]. \tag{92}$$

These are the well know RG equations for the KPZ equation [3–5].

In the analysis we made above, we discarded some terms that result from coarse graining the initial generating functional for the KPZ equation, the MN, and the CI terms. In principle, it is not difficult to take into account their effect in a systematical manner (for the Navier-Stokes equation, see Ref. [49]). However, provided $d>2$ we can see that both the MN term and the CI are irrelevant in the special case when we are near the trivial fixed point. This fixed point is given by $\lambda=0$, $z=2$, and $\alpha=1-d/2$. The MN term rescales as $b^{(2-d)}$, and the CI as $b^{(4-2d)}$. Hence, if d is greater than 2 both terms tends to zero exponentially when $s\to\infty$.

We mention here that the concept of renormalization group has also been fruitful in studying differential equations within singular perturbation theory [50]. In these papers, the renormalization is applied to the parameters appearing in the perturbative solutions of the differential equations, and not to the parameters of the differential equations themselves, as in the present work. As pointed out by Kunihiro [51], the essence of the method pioneered by Goldenfeld, Oono, and their collaborators, is to find an envelope curve for a uniparameter family of perturbative solutions, locally valid, of the differential equations, selected among the whole set of such solutions, which depend on a greater number of parameters. For an application of this method to field theory see, for example, Ref. [52].

VI. FINAL REMARKS

In this paper we accomplished the following two things.

(a) With respect to the theory of the nonequilibrium renormalization group, we showed that it is possible to derive a nontrivial renormalization group flow from a CTP action. This renormalization group is different from the usual in quantum field theory textbooks (see, for example, Ref. [53]) in that it describes nontrivial noise and dissipation. This regime has not been observed in earlier studies of the renormalization group from the CTP effective action [1]. In Ref. [1], the starting point was a noiseless, time reversal invariant theory, which was investigated within perturbation theory. But the relevant noise and dissipation effects are essentially nonperturbative [9]. A nontrivial nonequilibrium renormalization group can only be found in an ''environmentally friendly'' approach [26] where the basic description of the theory already has noise and dissipation built in.

(b) From the point of view of the renormalization group flow in the KPZ equation, we have derived the relevant flow equations from an analysis that consistently considered only the long wavelength sector of the theory. The usual approach of deriving these equations from the ultraviolet behavior of response functions [43], although technically correct, is conceptually contrived. Being explicitly dissipative, the KPZ equation should not be regarded as fundamental, but rather as the macroscopic limit of an underlying, unitary field theory, even if we lack a full specification of this microscopic description. In Ref. [43], the renormalization group flow is derived from a regime where the noisy and dissipative effective description embodied in the KPZ equation ceases to be valid, and the underlying unitary theory is recovered. The ultraviolet divergences in this underlying theory ought to be idenpendent of temperature, and therefore the same as in vacuum, leading to the usual ''textbook'' renormalization group [53]; we remark that by ''vacuum'' we mean the vacuum of the microscopic, unitary theory. For this reason we believe that the approach in the present paper, where no reference to ultraviolet behavior is made, is conceptually simpler, although technically equivalent.

One thing we did not accomplish is to describe in detail the crossover from the high-energy unitary theory to the low-energy noisy and dissipative effective theory. We bypassed this difficult problem by choosing as low-energy effective description a theory with a clear physical content. For example, if the high-energy theory leads to hydrodynamics in some regime, then it contains the Burgers and KPZ equation in the limit in which the pressure is null and the velocity field vorticity free. If we consider a theory of a scalar field, for example, we know this limit exists, because at low temperatures the field will behave as a condensate and develop a negative pressure, while at high temperatures the theory will be approximately conformally invariant, thus leading to a radiationlike equation of state. Thus the pressure will be much lower than the energy density at least in some intermediate range. The KPZ field, of course, is a collective mode when described in terms of the fundamental theory. We expect to continue our research on this issue.

The renormalization group as studied in this paper, is a necessary tool to understand the nature of collective variables describing the relevant physics in strongly interacting nonequilibrium systems such as the universe during the reheating period and the gluon fireball in the early stages of a high-energy heavy ion collision. We continue our research on this rewarding problem.

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APPENDIX A: AVERAGE OF THE LANGEVIN KPZ EQUATION

As a way of comparison with the results of Sec. III C, in this appendix we shall calculate the equation of motion for the mean (i.e., classical) field directly from the noisy KPZ equation (35). In momentum space it reads

$$[ip^0+\nu\vec{p}^2]\varphi(p)+\frac{\lambda}{8\pi^2}\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)\varphi(p_1)\times\varphi(p-p_1)=\eta(p). \tag{A1}$$

We write $\overline{\varphi}$ for the mean value of φ after averaging out the noise η, and define the fluctuating field ψ according to $\varphi=\overline{\varphi}+\psi$. Therefore, if we average the KPZ equation, it yields

$$[ip^0+\nu\vec{p}^2]\overline{\varphi}(p)+\frac{\lambda}{8\pi^2}\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)\times[\overline{\varphi}(p_1)\overline{\varphi}(p-p_1)+\langle\psi(p_1)\psi(p-p_1)\rangle]=0, \tag{A2}$$

and thus

$$[ip^0+\nu\vec{p}^2]\psi(p)+\frac{\lambda}{8\pi^2}\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)\times[2\psi(p_1)\overline{\varphi}(p-p_1)+\psi(p_1)\psi(p-p_1)-\langle\psi(p_1)\psi(p-p_1)\rangle]=\eta(p). \tag{A3}$$

We shall write the solution for ψ as a power series in λ,

$$\psi(p)=\psi^{(0)}(p)+\lambda\psi^{(1)}(p)+\cdots, \tag{A4}$$

from which the following expressions result:

$$\psi^{(0)}(p)=\frac{\eta(p)}{[ip^0+\nu\vec{p}^2]}, \tag{A5}$$

$$\psi^{(1)}(p)=-\frac{1}{4\pi^2[ip^0+\nu\vec{p}^2]}\times\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)[2\psi^{(0)}(p_1)\overline{\varphi}(p-p_1)+\psi^{(0)}(p_1)\psi^{(0)}(p-p_1)\langle\psi^{(0)}(p_1)\psi^{(0)}(p-p_1)\rangle]. \tag{A6}$$

Hence,

$$\langle\psi^{(0)}(p)\psi^{(0)}(p')\rangle=\frac{N(p,p')}{[ip^0+\nu\vec{p}^2][ip'^0+\nu\vec{p}'^2]}, \tag{A7}$$

$$\langle\psi^{(1)}(p)\psi^{(0)}(p')\rangle=\frac{-\lambda}{4\pi^2[ip^0+\nu\vec{p}^{\,2}]}\times\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)\times\frac{N(p_1,p')\overline{\varphi}(p-p_1)}{[ip_1^0+\nu\vec{p}_1^2][ip'^0+\nu\vec{p}'^2]}. \tag{A8}$$

Coming back to Eq. (A2) we find that

$$[ip^0+\nu\vec{p}^2]\overline{\varphi}(p)+\frac{\lambda}{8\pi^2}\int d^4p_1\vec{p}_1\cdot(\vec{p}-\vec{p}_1)\overline{\varphi}(p_1)\overline{\varphi}(p-p_1)+\frac{\lambda}{8\pi^2}\int d^4p_1\Delta_{ij}(p_1,p-p_1)\delta_{ij}-\frac{\lambda^2}{16\pi^4}\int d^4p_1d^4p_2\frac{\Delta_{ij}(p_2,p-p_1)(\vec{p}_1-\vec{p}_2)_i\vec{p}_{1_j}}{[ip_1^0+\nu\vec{p}_1^2]}\times\overline{\varphi}(p_1-p_2)=0, \tag{A9}$$

where Δ is defined in Eq. (60). The change $p_1\to -p_1+p$ yields to the same expression we found by computing the CTP EA, Eq. (61), when $j=0$.

APPENDIX B

In this appendix we evaluate Eq. (73) to order λ^2. We start from Eq. (73). The average of an odd number of $>$ fields is zero, because of the parity of the free action (71) when we change $\varphi_>$ by $-\varphi_>$ and, simultaneously, $\phi_>$ by $-\phi_>$. Hence, up to $O(\lambda^2)$, we find

$$\Delta S[\phi_<,\varphi_<]=-i\Bigg\{\frac{i\lambda}{2}\int\mathcal{D}\phi_>\mathcal{D}\varphi_>e^{iS_0[\phi_>,\varphi_>]}\Bigg[\int dp_{123}\vec{p}_2\cdot\vec{p}_3\{\phi_{<1}\varphi_{>2}\varphi_{>3}+2\phi_{>1}\varphi_{>2}\varphi_{<3}\}\Bigg]-\frac{\lambda^2}{8}\int\mathcal{D}\phi_>\mathcal{D}\varphi_>e^{iS_0[\phi_>,\varphi_>]}\Bigg[\int dp_{123}dq_{123}\vec{p}_2\cdot\vec{p}_3\vec{q}_2\cdot\vec{q}_3\times\{\phi_{>1}\varphi_{>2}\varphi_{>3}\phi_{>\tilde{1}}\varphi_{>\tilde{2}}\varphi_{>\tilde{3}}+\phi_{<1}\phi_{<\tilde{1}}\varphi_{>2}\varphi_{>3}\varphi_{>\tilde{2}}\varphi_{>\tilde{3}}+4\phi_{<1}\varphi_{<2}\phi_{<\tilde{1}}\varphi_{<\tilde{2}}\varphi_{>3}\varphi_{>\tilde{3}}+4\varphi_{<3}\varphi_{<\tilde{3}}\phi_{>1}\varphi_{>2}\phi_{>\tilde{1}}\varphi_{>\tilde{2}}+\varphi_{<2}\varphi_{<3}\varphi_{<\tilde{2}}\varphi_{<\tilde{3}}\phi_{>1}\phi_{>\tilde{1}}+4\phi_{<1}\varphi_{<2}\phi_{>\tilde{1}}\varphi_{>\tilde{2}}\varphi_{>\tilde{3}}\varphi_{>3}+2\varphi_{<2}\varphi_{<3}\phi_{>\tilde{1}}\varphi_{>\tilde{2}}\varphi_{>\tilde{3}}\phi_{>1}+4\phi_{<1}\varphi_{<\tilde{3}}\varphi_{>2}\varphi_{>3}\phi_{>\tilde{1}}\varphi_{>\tilde{2}}+4\phi_{<1}\varphi_{<2}\varphi_{<\tilde{2}}\varphi_{<\tilde{3}}\varphi_{>3}\phi_{>\tilde{1}}\}\Bigg]\Bigg\}_{connected}. \tag{B1}$$

Here, $\varphi_{<i}$ ($\varphi_{<\tilde{i}}$) means $\varphi_<(p_i)$ ($\varphi_<(q_i)$); dp_{123} stands for $d^4p_1d^4p_2d^4p3\,\delta(p_1+p_2+p_3)$ and analogously for dq_{123}. In computing the last expression, as the logarithm of Eq. (73) is taken, we must discard the disconnected diagrams associated with each functional integration. Also, we must have in mind that when some integration variable, say p_1, satisfies $|\vec{p}_1|<\Lambda_s$, then $\varphi_>(p_1)$ and $\phi_>(p_1)$ will vanish; conversely if $|\vec{p}_1|>\Lambda_s$. To evaluate Eq. (B1) we can employ the propagators given in Eq. (51) with slightly modifications, that is,

$$\langle\phi_>(p)\phi_>(p')\rangle=0, \tag{B2}$$

$$\langle\varphi_>(p)\varphi_>(p')\rangle=\frac{N(p,p')}{[ip^0+\nu\vec{p}^2][ip'^0+\nu\vec{p}'^2]},$$

$$\langle\phi_>(p)\varphi_>(p')\rangle=\frac{i\delta(p+p')}{[ip'^0+\nu\vec{p}'^2]}.$$

It is understood that if some momentum in the previous equations lies below Λ_s, the corresponding propagator will be null. In Fig. 1 we give the convention adopted to represent the propagators listed above. These propagators will be used as internal lines in Feynman diagrams. When computing

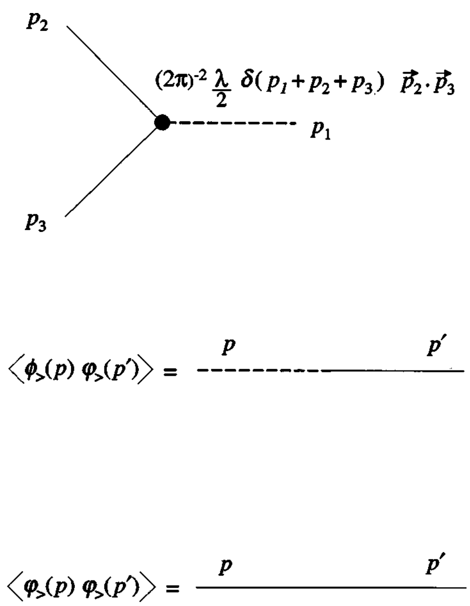

FIG. 1. Vertex and propagators used to calculate the coarse grained action for the KPZ equation.

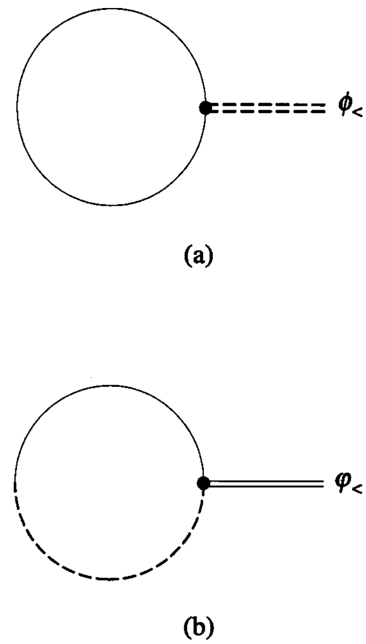

FIG. 2. Order λ Feynman diagrams for the coarse grained action. The external fields are indicated by double lines.

these diagrams, for each vertex there will be an integral over the three momenta attached to it. After splitting the fields, the propagators quoted in Eq. (50) are not valid any longer. However, because the split is in wavelength and not in frequency, the causal properties of the propagators are still the same.

Consider the terms of order λ in Eq. (B1), the first of which adds to the action a term that is (functionally) linear in $\phi_<$,

$$-\frac{\lambda}{2}\int d^4p(2\pi)^{-2}\delta(p)\phi_<(p)\times\left\{\int_{\Lambda_s<|\vec{q}|<\Lambda} d^4q\frac{N(q,-q)}{\delta(0)}\frac{\vec{q}^{\,2}}{[(q^0)^2+\nu^2(\vec{q}^{\,2})^2]}\right\}, \tag{B3}$$

where we have assumed that the noise represented by N is not only TI but also white. Diagrammatically this term is shown in Fig. 2(a). The external lines take trace of the $<$ fields that are attached to a given vertex: a double continuous line represents a $\varphi_<$ field; a double dashed line is used for a $\phi_<$ field. The contribution given in Eq. (B3), in turn, can be seen as a field-independent term added to the classical KPZ equation (A1). The remaining term of $O(\lambda)$, represented in Fig. 2(b), is zero. This is because the propagator $\langle\phi_>(p_1)\varphi_>(p_2)\rangle$ introduces a $\delta(p_1+p_2)$. In addition to the conservation delta, $\delta(p_1+p_2+p_3)$, it implies a $\delta(p_3)$. The product of this $\delta(p_3)$ with $\vec{p}_2\cdot\vec{p}_3$ force the whole integral to vanish. We now proceed to consider the terms of order λ^2 in Eq. (B1).

(a) The first one corresponds to three connected, nonequivalent diagrams [Fig. 3(a)], and gives a contribution that does not depend on $\varphi_<$ or $\phi_<$. However, each of these diagrams is zero, either because they entail a delta evaluated in a momentum that lies outside the integration domain, or a product of two mutually excluding θ's. We shall find more of these cancellations below.

(b) The second term of order λ^2 [Fig. 3(b)], consists on a closed loop, and has the same structure as the noise term in the original action. Explicitly, this diagram gives the following contribution to the CGA action:

$$\frac{i\lambda^2}{4}\int d^4p_1d^4q_1\phi_{<1}\phi_{<\tilde{1}}\left\{\int d^4p_2d^4p_3d^4q_2d^4q_3(2\pi)^{-4}\delta_{123}\delta_{\tilde{1}\tilde{2}\tilde{3}}\vec{p}_2\cdot\vec{p}_3\vec{q}_2\cdot\vec{q}_3\times\frac{N(p_2,q_2)N(p_3,q_3)}{(ip_2^0+\nu\vec{p}_2^{\,2})(iq_2^0+\nu\vec{q}_2^{\,2})(ip_3^0+\nu\vec{p}_3^{\,2})(iq_3^0+\nu\vec{q}_3^{\,2})}\mathbf{M}(p_2,p_3,q_2,q_3)\right\}, \tag{B4}$$

where δ_{123} stands for $\delta(p_1+p_2+p_3)$ and $\delta_{\tilde{1}\tilde{2}\tilde{3}}$ for $\delta(q_1+q_2+q_3)$. $\mathbf{M}$ is the product of the projectors over each three-momentum shell of its arguments, and is inserted to take trace of the proper integration domains, that is,

$$\mathbf{M}(\{p_i\})=\prod_i\theta(|\vec{p}_i|-\Lambda_s)\theta(\Lambda-|\vec{p}_i|). \tag{B5}$$

Up to this point, we assumed that the noise was zero mean Gaussian, white, and TI. For the sake of simplicity, we now assume that the noise represented by N does not have spatial correlations as well, so that it satisfies Eq. (74). Equation (B4) now reads

$$\frac{i}{2}\int d^4p\, \phi_<(-p)\left\{2D^2\lambda^2\int \frac{d^4q}{4\pi^2}\frac{[\vec{q}\cdot(\vec{p}-\vec{q})]^2\mathbf{M}(q,p-q)}{[(q^0)^2+\nu^2(\vec{q}^2)^2][(p^0-q^0)^2+\nu^2\{(\vec{p}-\vec{q})^2\}^2]}\right\}\phi_<(p). \tag{B6}$$

(c) The third term of $O(\lambda^2)$ in Eq. (B1) adds to the original action a new term, not included previously [Fig. 3(c)]. The contribution of this term to the exponenciated CGA can be written as

$$e^{i\Delta S_{3\mathrm{rd}}}=\exp\left\{-\lambda^2D\int d^4q\left(\int\frac{d^4p}{4\pi^2}\phi_<(-p)\varphi_<(p-q)\frac{(\vec{p}-\vec{q})\cdot\vec{q}}{(iq^0+\nu\vec{q}^2)}\mathbf{M}(q)\right)\right.$$
$$\left.\times\left(\int\frac{d^4p}{4\pi^2}\phi_<(-p)\varphi_<(p+q)\frac{[-(\vec{p}+\vec{q})\cdot\vec{q}]}{(-iq^0+\nu\vec{q}^2)}\mathbf{M}(-q)\right)\right\}. \tag{B7}$$

In turn, we can regard this contribution as coming from a new source of noise, in a sense that will become clear after we express $e^{i\Delta S_{3\mathrm{rd}}}$ as the functional Fourier transform of an appropriate expression [34,2,9,35,36,24]. That is,

$$e^{i\Delta S_{3\mathrm{rd}}}=\mathcal{Z}\int\mathcal{D}\rho\exp\left(-[4\lambda^2D]^{-1}\int d^4q\,\rho(q)\rho(-q)\right)\exp\left\{i(2\pi)^{-2}\int d^4q\,d^4p\,\rho(q)\phi_<(-p)\varphi_<(p-q)\frac{(\vec{p}-\vec{q})\cdot\vec{q}}{(iq^0+\nu\vec{q}^2)}\mathbf{M}(q)\right\}, \tag{B8}$$

where $\mathcal{Z}$ is a normalization factor. Hence, $e^{i\Delta S_{3\mathrm{rd}}}$ can be seen as the average of certain new term, according to the probability distribution of the auxiliary source ρ. This distribution is that of a white, TI, and Gaussian noise, which has a second order momentum equal to $2D\lambda^2$. Moreover, ρ is a multiplicative, rather than an additive noise.

(d) The fourth term of order λ^2 in Eq. (B1), is represented as a one loop diagram, built up by two propagators $\langle\phi_>\varphi_>\rangle$ [Fig. 3(d)]. When calculating this propagator in the coordinate representation, $\langle\phi_>(x)\varphi_>(x')\rangle$, the separation on lower and higher (spatial) wave numbers modes, does not prevent the arising of a $\theta(t'-t)$, as result of integrating p^0 in the complex plane when the Fourier transform of $\langle\phi_>(p)\varphi_>(p')\rangle$ is performed in reverse. Thus, the double product $\langle\phi_>(p')\varphi_>(p)\rangle \times\langle\phi_>(p)\varphi_>(p')\rangle$ has null-measure support, and the diagram vanishes. (The fifth term is proportional to the propagator of two fields $\phi_>$, which is zero.)

(e) The sixth term is the sum of two nonequivalent diagrams [Fig. 3(e)],

$$\frac{i\lambda^2}{2}\int d^4p_1d^4p_2\phi_{<1}\varphi_{<2}\left\{\int d^4p_3d^4q_1d^4q_2d^4q_3(2\pi)^{-4}\delta_{123}\delta_{\tilde{1}\tilde{2}\tilde{3}}\vec{p}_2\cdot\vec{p}_3\vec{q}_2\cdot\vec{q}_3\right.$$
$$\left.\times\left[2\frac{i\delta(q_1+q_2)}{(iq_2^0+\nu\vec{q}_2^2)}\frac{N(p_3,q_3)}{(ip_3^0+\nu\vec{p}_3^2)(iq_3^0+\nu\vec{q}_3^2)}+\frac{N(q_2,q_3)}{(iq_2^0+\nu\vec{q}_2^2)(iq_3^0+\nu\vec{q}_3^2)}\frac{i\delta(q_1+p_3)}{(ip_3^0+\nu\vec{p}_3^2)}\right]\mathbf{M}(p_3,q_1,q_2,q_3)\right\}, \tag{B9}$$

The first delta function in the square brackets gives a $\delta(q_3)$, but $\vec{q}_3$ must be integrated in a shell that does not include the origin, and therefore the contribution of this member vanishes. For similar reasons, because $N(p,p')\propto\delta(\vec{p}+\vec{p}')$, the second member in the square brackets also vanishes.

(f) The seventh term of order λ^2 in Eq. (B1) is represented by a single diagram [Fig. 3(f)], which includes a loop formed by the propagator $\langle\phi_{>\tilde{1}}\varphi_{>\tilde{2}}\rangle\propto\delta(q_1+q_2)$. This delta function and that of conservation $\delta_{\tilde{1}\tilde{2}\tilde{3}}$, generate a $\delta(q_3)$. Because of the integration domain, as before, the diagram gives no contributions.

(g) The eighth term [Fig. 3(g)] gives the following contribution to the CGA:

$$-\lambda^2\int d^4p_1\phi_<(-p_1)\left\{(2\pi)^{-4}\int d^4p_3d^4q_3\vec{p}_3\cdot(\vec{p}_1-\vec{p}_3)\vec{q}_3\cdot(\vec{p}_3-\vec{q}_3)\right.$$
$$\left.\times\frac{N(p_1-p_3,p_3-q_3)\varphi_<(q_3)}{(ip_3^0+\nu\vec{p}_3^2)(i[p_1-p_3]^0+\nu[\vec{p}_1-\vec{p}_3]^2)(i[p_3-q_3]^0+\nu[\vec{p}_3-\vec{q}_3]^2)}\mathbf{M}(p_3,p_1-p_3,p_3-q_3)\right\}. \tag{B10}$$

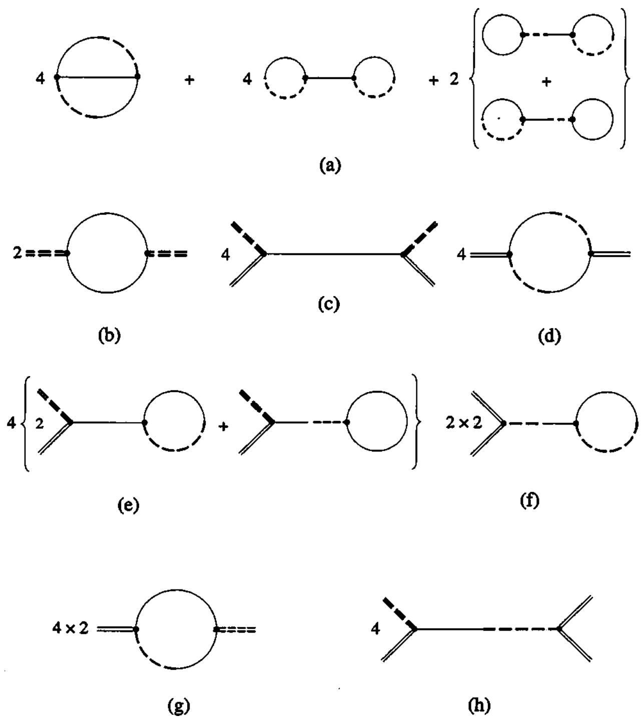

FIG. 3. Order λ^2 Feynman diagrams for the coarse grained action.

This contribution can be thought of as a momentum dependent correction to the viscosity term in Eq. (A1). When expanded in powers of the external momentum p_1, the curly bracket takes the form of an infinite sum of derivative interactions. We saw in Sec. V that, if the noise satisfies Eq. (74), in the limit in which the shell is made of infinitesimal thickness, the expression between curly brackets gives, for small external p_1, a factor proportional to $\vec{p}_1^{\,2}$. All other contributions are of higher order in $|\vec{p}_1|$.

(h) Finally, the ninth term of order λ^2 in Eq. (B1), generates a new vertex (the cubic interaction term), which couples three $\varphi_<$'s with one $\phi_<$ [Fig. 3(h)],

$$-\frac{\lambda^2}{2}\int d^4p_1 d^4p_2 d^4q_2 d^4q_3 (2\pi)^{-4}\delta_{12\tilde{2}\tilde{3}} \times \frac{\vec{p}_2\cdot(\vec{q}_2+\vec{q}_3)\vec{q}_2\cdot\vec{q}_3\mathbf{M}(q_2+q_3)}{[-i(p_1^0+p_2^0)+\nu(\vec{p}_1+\vec{p}_2)^2]}\phi_{<1}\varphi_{<2}\varphi_{<\tilde{2}}\varphi_{<\tilde{3}}. \tag{B11}$$

APPENDIX C

In this appendix we compare the results of the preceding section with those obtained by coarse graining the equations of motion. This is the first step of the transformation associated with the dynamical RG as defined in Ref. [6], to be further discussed below.

1. Definitions

In general, we start from a given stochastic equation

$$\mathcal{L}\{\varphi\}(p)+\mathcal{N}\{\varphi\}(p)=\eta(p), \tag{C1}$$

where the operator $\mathcal{L}$ is linear and $\mathcal{N}$ collects the nonlinear terms, and where, to be specific, the noise η verifies Eq. (74). The nonlinearity couples modes of different scales. An exact solution can be attained in few cases only, such as the noiseless Burger's equation in $1+1$ dimensions [48]. One could be interested in reducing the number of modes—for a computational calculation on a discrete lattice [45,49]—or in studying the scaling properties—in relation with critical phenomena [6]. In both cases the elimination of short scale modes can be accomplished by solving their equations of motion in terms of the long scale modes, adopting, in general, some perturbative scheme. One then feeds back these solutions in the equations of motion of the long scale modes, obtaining a coupled set of effective equations for these modes only. One identifies, in these equations, effective couplings—some of which were zero in the initial equations—and noise terms, which can be either additive or multiplicative.

Formally, we can define a projector $\mathcal{P}$ over the Fourier space spanned by modes in the momentum shell $\Lambda_s<|\vec{p}|<\Lambda$, and project the Eq. (C1) to obtain

$$\mathcal{L}\{\varphi_>\}+\mathcal{P}\mathcal{N}\{\varphi_>+\varphi_<\}=\eta_>, \tag{C2}$$

$$\mathcal{L}\{\varphi_<\}+(1-\mathcal{P})\mathcal{N}\{\varphi_>+\varphi_<\}=\eta_<. \tag{C3}$$

In some way we must solve the first equation for $\varphi_>$ to obtain $\varphi_>[\varphi_<,\eta_>]$. The second equation is then rewritten as

$$\mathcal{L}\{\varphi_<\}+(1-\mathcal{P})\mathcal{N}\{\varphi_<+\varphi_>[\varphi_<,\eta_>]\}=\eta_<. \tag{C4}$$

This will be the effective equation for the long modes, and the one we expect that reproduces the results obtained from coarse graining the CTP generating functional. Some fluctuating terms on the left-hand side of Eq. (C4) can be added to the noise $\eta_<$ to form an effective noise $\tilde{\eta}_<$, which will have an amplitude (or more precisely, a two point correlation function characterized by an amplitude) $\tilde{D}$. We remark that in Eq. (C4) there is not implicit any kind of averaging process. The effective noise amplitude can be obtained trivially from Eq. (C4) by calculating the correlation of $\tilde{\eta}_<$.

This situation is different from that addressed, for example, in the paper of Medina *et al.* [4], concerning the KPZ equation, where the effective noise amplitude is derived from the two point correlation function of the fields, or that presented by McComb for the Navier-Stokes equation in Ref. [45], where the effective equation is averaged with respect to the short scale noise. For example, the right-hand side of Eq. (9.34) in McComb's book [45] displays the unrenormalized external force, while our approach would replace it by the effective one [see Eqs. (3.11) and (3.18) of Ref. [44]]. This difference arises because of the way the average is performed in Eq. (9.16) of Ref. [45].

We show below the results of applying the coarse graining procedure to the KPZ equation.

2. Coarse graining of KPZ equation

Starting from Eq. (A1), we proceed as before, splitting the field as the sum of two independent fields, $\varphi=\varphi_>+\varphi_<$, and analogously for the noise η. Thus,

$$(ip_0+\nu\vec{p}^2)[\varphi_>+\varphi_<](p) + \frac{\lambda}{2}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\{\varphi_>(q)\varphi_>(p-q) + 2\varphi_>(q)\varphi_<(p-q)+\varphi_<(q)\varphi_<(p-q)\} = [\eta_>+\eta_<](p). \tag{C5}$$

If $|\vec{p}|>\Lambda_s$, then $\varphi_<(p)$ and $\eta_<(p)$ are zero, and we obtain

$$(ip_0+\nu\vec{p}^2)\varphi_>(p) + \frac{\lambda}{2}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\{\varphi_>(q)\varphi_>(p-q) + 2\varphi_>(q)\varphi_<(p-q)+\varphi_<(q)\varphi_<(p-q)\} = \eta_>(p). \tag{C6}$$

This equation can be solved, formally, order by order in λ by setting

$$\varphi_>=\varphi_>{}^{(0)}+\lambda\varphi_>{}^{(1)}+\cdots. \tag{C7}$$

It yields

$$\varphi_>{}^{(0)}(p)=\frac{\eta_>(p)}{(ip^0+\nu\vec{p}^2)}, \tag{C8}$$

$$\varphi_>{}^{(1)}(p)=\frac{-1}{2(ip^0+\nu\vec{p}^2)}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q}) \times\left\{\frac{\eta_>(q)}{(iq^0+\nu\vec{q}^2)}\frac{\eta_>(p-q)}{(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)} + 2\varphi_<(q)\frac{\eta_>(p-q)}{(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)} + \varphi_<(q)\varphi_<(p-q)\right\}. \tag{C9}$$

When $|\vec{p}|<\Lambda_s$, $\varphi_>(p)$ and $\eta_>(p)$ are zero. For such p, and using the expressions given in Eqs. (C8) and (C9), we find a closed equation for the field $\varphi_<$, namely,

$$\begin{aligned}
&(ip^0+\nu\vec{p}^2)\varphi_<(p)+\frac{\lambda}{2}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\left[\varphi_<(q)\varphi_<(p-q)+2\varphi_<(p-q)\frac{\eta_>(q)}{(iq^0+\nu\vec{q}^2)}+\frac{\eta_>(q)\eta_>(p-q)}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)}\right]\\
&-\frac{\lambda^2}{2}\int\frac{d^4q\,d^4k}{16\pi^4}\vec{q}\cdot(\vec{p}-\vec{q})\vec{k}\cdot(\vec{q}-\vec{k})\mathbf{M}(q)\left[\frac{\varphi_<(p-q)\varphi_<(k)\varphi_<(q-k)}{(iq^0+\nu\vec{q}^2)}+2\frac{\varphi_<(p-q)\varphi_<(q-k)\eta_>(k)}{(iq^0+\nu\vec{q}^2)(ik^0+\nu\vec{k}^2)}\right.\\
&+\frac{\varphi_<(p-q)\eta_>(k)\eta_>(q-k)}{(iq^0+\nu\vec{q}^2)(ik^0+\nu\vec{k}^2)(i[q-k]^0+\nu[\vec{q}-\vec{k}]^2)}+\frac{\varphi_<(k)\varphi_<(q-k)\eta_>(p-q)}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)}\\
&+2\frac{\varphi_<(q-k)\eta_>(k)\eta_>(p-q)}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)(ik^0+\nu\vec{k}^2)}\\
&\left.+\frac{\eta_>(k)\eta_>(q-k)\eta_>(p-q)}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)(ik^0+\nu\vec{k}^2)(i[q-k]^0+\nu[\vec{q}-\vec{k}]^2)}\right]=\eta_<(p).
\end{aligned} \tag{C10}$$

This is the basic result: an effective equation that only contains the long modes $\varphi_<$.

We now re-sort things in order to clarify the meaning of each term. The first term of $O(\lambda)$ is the original nonlinearity. In the second one, $\eta_>$ acts like a multiplicative noise over $\varphi_<$, and can be identified with what we found earlier in Eq. (B8). Rewrite the third term of $O(\lambda)$ in Eq. (C10) as

$$\frac{\lambda}{2}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\left[\frac{N(q,p-q)_>}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)} + \left\{\frac{\eta_>(q)\eta_>(p-q)-N(q,p-q)_>}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)}\right\}\right]. \tag{C11}$$

With $N_>$ we have indicated that the functions N are zero if their arguments lie outside the momentum shell. In this expression, the first term gives a field independent contribution, which when the noise is delta correlated reproduces the result shown by Eq. (B3). In its turn, the remaining term in Eq. (C11) is an additive source of noise, and therefore the effective noise term, to $O(\lambda)$, is given by

$$\eta_<(p)-\frac{\lambda}{2}\int\frac{d^4q}{4\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\times\left[\frac{\eta_>(q)\,\eta_>(p-q)-N(q,p-q)_>}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)}\right]. \tag{C12}$$

This noise has zero mean, and its two point correlation function, assuming η satisfies Eq. (74), is given by

$$2D+2D^2\lambda^2\int\frac{d^4q}{4\pi^2}\times\frac{[\vec{q}\cdot(\vec{p}-\vec{q})]^2\mathbf{M}(q,p-q)}{[(q^0)^2+\nu^2(\vec{q}^2)^2][(p^0-q^0)^2+\nu^2\{(\vec{p}-\vec{q})^2\}^2]}. \tag{C13}$$

Notice that the correction to the noise two point correlation introduced above is equal to that given in Eq. (B6). However, the third order correlation function is not zero, so we cannot say that the effective noise is Gaussian, as was the original one. [However, the third order momentum is $O(\lambda^3)$]. To say something about the higher order correlation functions first we must include corrections to the noise coming from higher order terms in the perturbative expansion we performed to arrive at Eq. (C10).

Consider the $O(\lambda^2)$ terms in Eq. (C10). The first one is just the cubic interaction given in Eq. (B11). The second and the fourth terms are quadratic interactions subjected to multiplicative noise. The third term contains also multiplicative noise which, when the noise η is TI, has zero mean [because of $\mathbf{M}(q)$, and, hence, it does not contribute to the effective viscosity]. In the CGA all these multiplicative noisy terms appear when the perturbative expansion is extended to $O(\lambda^4)$.

The fifth term $O(\lambda^2)$ can be written as

$$-\frac{\lambda^2}{2}\int\frac{d^4q\,d^4k}{16\pi^2}\vec{q}\cdot(\vec{p}-\vec{q})\vec{k}\cdot(\vec{q}-\vec{k})\times\left[\frac{2\varphi_<(q-k)N(k,p-q)_>}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)(ik^0+\nu\vec{k}^2)}+\frac{2\varphi_<(q-k)\{\eta_>(k)\,\eta_>(p-q)-N(k,p-q)_>\}}{(iq^0+\nu\vec{q}^2)(i[p-q]^0+\nu[\vec{p}-\vec{q}]^2)(ik^0+\nu\vec{k}^2)}\right]. \tag{C14}$$

Hence, we can regard the first contribution as the momentum dependent correction to the viscosity we found in Eq. (B10), and the second one as another term linear in $\varphi_<$ subjected to multiplicative noise. This term, as with the last one appearing in Eq. (C10), which contributes to the effective noise, is found at $O(\lambda^4)$ in the CGA. We conclude that the coarse grained equation of motion coincides with the equation of motion derived from the CGA.

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